

# Anisotropic harmonic oscillator, non-commutative Landau problem and exotic Newton-Hooke symmetry

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## Abstract

We investigate the planar anisotropic harmonic oscillator with explicit rotational symmetry as a particle model with non-commutative coordinates. It includes the exotic Newton-Hooke particle and the non-commutative Landau problem as special, isotropic and maximally anisotropic, cases. The system is described by the same (2+1)-dimensional exotic Newton-Hooke symmetry as in the isotropic case, and develops three different phases depending on the values of the two central charges. The special cases of the exotic Newton-Hooke particle and non-commutative Landau problem are shown to be characterized by additional,  $so(3)$  or  $so(2,1)$  Lie symmetry, which reflects their peculiar spectral properties.

## 1 Introduction

Classical and quantum theories in 2+1 dimensions possess various exotic properties. These include, in particular, a possibility for existence of particles with fractional spin and statistics – anyons. Another peculiar property is an equivalence of a classical (2+1)-dimensional pure gravity to a Chern-Simons gauge theory.

In a special non-relativistic limit, that is an Inönü-Wigner contraction, (2+1)D Poincaré symmetry of a free anyon theory is reduced to an exotic Galilei symmetry with two central charges [1, 2, 3, 4, 5, 6, 7, 8]. A similar limit applied to the  $AdS_3$ , that is an asymptotic symmetry of the BTZ black hole solution of the 3D pure gravity [10], produces an exotic Newton-Hooke (ENH) symmetry with two central charges [11, 12, 13]. Both exotic, Galilei and Newton-Hooke, symmetries can be realized as symmetries of a particle on a non-commutative plane. The latter symmetry is transformed into the former one in a flat limit. The two-fold central extensions of the Galilei and Newton-Hooke symmetries are possible only in 2+1 dimensions<sup>1</sup>.

Like the BTZ black hole solution [10], a particle system with (2+1)D exotic Newton-Hooke symmetry displays three different phases in dependence on the values of the model parameters [13]. On the other hand, its reduced phase space description reveals a symplectic structure similar

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<sup>1</sup>The case of an exotic non-relativistic string in 3+1 dimensions and the relation with the exotic particle in 2+1 dimensions has been recently studied in [9].

to that of Landau problem in the non-commutative plane [5, 14, 15]. The noncommutative Landau problem (NLP) also develops three phases, the sub- and super-critical ones, separated by a critical, quantum Hall effect phase [15]. Therefore, these similarities indicate on a possible close relation between the (2+1)D exotic Newton-Hooke symmetry and the non-commutative Landau problem. The purpose of this article is to study in detail this relation by means of a planar exotic *anisotropic* harmonic oscillator with explicit *spatial rotation* symmetry as a particle model with non-commutative coordinates.

The model of the anisotropic harmonic oscillator we propose [(2.2) below], includes the exotic Newton-Hooke particle and the non-commutative Landau problem as special, isotropic and maximally anisotropic, cases. We show that what distinguishes the exotic Newton-Hooke particle and non-commutative Landau problem as special cases, is a presence of the additional,  $so(3)$  or  $so(2,1)$  Lie symmetry. In a generic case of commensurable frequencies, the exotic anisotropic harmonic oscillator is characterized, instead, by a nonlinear deformation of the indicated additional Lie symmetry. Like the exotic Newton-Hooke particle and non-commutative Landau problem, the anisotropic oscillator system develops the subcritical and supercritical phases, separated by a critical phase. The phase is defined by the values of the two central charges of the exotic Newton-Hooke algebra.

The paper is organized as follows. In Section 2 we introduce a planar anisotropic harmonic oscillator with explicit rotational symmetry as a particle model with non-commutative coordinates, and establish its relation with the non-commutative Landau problem. In Section 3 we discuss the chiral form of the exotic Newton-Hooke symmetry of the system, and analyze its additional symmetries, which depend on the concrete values of the model parameters. In Section 4 we analyze the exotic Newton-Hooke symmetry in the non-chiral, space-time picture. Section 5 is devoted to the discussion and concluding remarks.

## 2 Planar anisotropic harmonic oscillator and non-commutative Landau problem

A canonical Lagrangian of one dimensional harmonic oscillator of mass  $m$  and frequency  $\omega$  is given by

$$L_{can} = \frac{\mu}{2} \left( \epsilon_{ij} \dot{X}_i X_j - \frac{\alpha}{R} X_i^2 \right), \quad (2.1)$$

where  $m = \alpha^{-1} \mu R$ ,  $\omega = \alpha R^{-1}$ , and  $\alpha$  is a dimensionless parameter. Variable  $X_1$  can be identified as the coordinate of one dimensional particle, and then  $X_2$  is proportional to its momentum. Symplectic structure,  $\{X_i, X_j\} = \frac{1}{\mu} \epsilon_{ij}$ , and Lagrangian (2.1) possess a two dimensional *phase space* rotational symmetry. Taking a sum of  $n$  copies of (2.1) with independent parameters  $\mu$ 's and  $\alpha$ 's, we obtain a generalized system of  $n$  non-interacting harmonic oscillators with different frequencies.

Let us consider the case  $n = 2$ , and take the canonical Lagrangian in the form

$$L_{+-} = -\frac{\mu_+}{2} \left( \epsilon_{ij} \dot{X}_i^+ X_j^+ + \frac{\alpha_+}{R} X_i^{+2} \right) - \frac{\mu_-}{2} \left( -\epsilon_{ij} \dot{X}_i^- X_j^- + \frac{\alpha_-}{R} X_i^{-2} \right). \quad (2.2)$$

We suppose that  $R > 0$  and that  $\mu_{\pm}$  can take values of any sign. For the moment we do not assume any restrictions for the parameters  $\alpha_{\pm}$ . The dynamics of (2.2) is given by

$$\dot{X}_i^{\pm} \pm \omega_{\pm} \epsilon_{ij} X_j^{\pm} = 0. \quad \omega_{\pm} = \alpha_{\pm} R^{-1}, \quad (2.3)$$

while its symplectic structure is

$$\{X_i^+, X_j^+\} = -\frac{1}{\mu_+} \epsilon_{ij}, \quad \{X_i^-, X_j^-\} = \frac{1}{\mu_-} \epsilon_{ij}, \quad \{X_i^+, X_j^-\} = 0. \quad (2.4)$$

In the chosen special case  $n = 2$ , a *phase space* index  $i$  can be reinterpreted as a *spatial* index of the (2+1)-dimensional space-time. With such reinterpretation, Lagrangian (2.2) as well as equations of motion (2.3) and symplectic structure (2.4) possess the explicit *spatial*  $SO(2)$  rotation symmetry. This corresponds to the diagonal part of the obvious chiral rotation symmetry  $SO(2) \times SO(2)$  of (2.2). The nondiagonal part is identified with the time translation symmetry, see Eqs. (3.9) and (3.10) below.

As a result, system (2.2) provides us with a *rotational invariant* description of the planar *anisotropic* harmonic oscillator system.

At this point we would like to clarify under which conditions this planar anisotropic oscillator can be interpreted as a two dimensional *particle* system with coordinate  $x_i$  and velocity  $v_i$  related by a usual dynamics equation

$$\dot{x}_i = v_i. \quad (2.5)$$

In accordance with (2.3), for such a particle system the variables  $X_i^+$  and  $X_i^-$  should have a sense of normal, or chiral modes.

To this aim, we first note that the transformation  $\alpha_{\pm} \rightarrow -\alpha_{\mp}$ ,  $\mu_{\pm} \rightarrow -\mu_{\mp}$ ,  $X_i^+ \leftrightarrow X_i^-$  does not change equations of motion and the Lagrangian, and it is sufficient to assume that  $(\alpha_+ + \alpha_-) \geq 0$ . If  $(\alpha_+ + \alpha_-) = 0$ , the two chiral modes have exactly the same evolution. This case should be excluded since it does not allow us to introduce  $x_i$  and  $v_i$  related by (2.5). Taking also into account that  $\alpha_{\pm}$  appear only in the combination  $\alpha_{\pm}/R$ , without loss of generality we can put

$$\alpha_{\pm}(\chi) = \cos \chi (\cos \chi \pm \sin \chi), \quad -\frac{\pi}{2} < \chi < \frac{\pi}{2}. \quad (2.6)$$

With the normalization chosen in (2.6), we have<sup>2</sup>  $0 < (\alpha_+ + \alpha_-) \leq 1$  and  $-1 \leq (\alpha_+ - \alpha_-) \leq 1$ .

Now we can introduce the coordinate and velocity of the two dimensional particle system,

$$X_i^{\pm} = \alpha_{\mp} x_i \pm R \epsilon_{ij} v_j, \quad (2.7)$$

$$x_i = \frac{1}{\alpha_+ + \alpha_-} (X_i^+ + X_i^-), \quad v_i = \frac{1}{R(\alpha_+ + \alpha_-)} \epsilon_{ij} (\alpha_- X_j^- - \alpha_+ X_j^+), \quad (2.8)$$

which satisfy dynamics relation (2.5). In a generic case, in correspondence with (2.4), the coordinate  $x_i$  describes a *non-commutative* plane,

$$\{x_i, x_j\} = \frac{1}{(\alpha_+ + \alpha_-)^2} \frac{\mu_+ - \mu_-}{\mu_+ \mu_-} \epsilon_{ij}. \quad (2.9)$$

The components of the coordinate vector  $x_i$  are commutative only when  $\mu_+ = \mu_-$ . As we shall see, only in this case the Galilean boosts mutually commute. Other brackets are

$$\{x_i, v_j\} = \frac{1}{R^2(\alpha_+ + \alpha_-)^2} \frac{\mathcal{M}}{\mu_+ \mu_-} \delta_{ij}, \quad \{v_i, v_j\} = \frac{1}{R^2(\alpha_+ + \alpha_-)^2} \frac{\mathcal{B}}{\mu_+ \mu_-} \epsilon_{ij}, \quad (2.10)$$

where

$$\mathcal{M} = R(\mu_+ \alpha_- + \mu_- \alpha_+), \quad \mathcal{B} = \mu_+ \alpha_-^2 - \mu_- \alpha_+^2. \quad (2.11)$$

Symplectic two-form corresponding to (2.9) and (2.10) has a simple structure,

$$\sigma = \mathcal{M} dv_i \wedge dx_i + \frac{1}{2} R^2 (\mu_+ - \mu_-) \epsilon_{ij} dv_i \wedge dv_j + \frac{1}{2} \mathcal{B} \epsilon_{ij} dx_i \wedge dx_j. \quad (2.12)$$

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<sup>2</sup> In what follows it will be more convenient to work, however, in terms of  $\alpha_{\pm}$ , implying the one-parametric representation (2.6).

In terms of the particle coordinate  $x_i$ , the anisotropy reveals itself in the coupled dynamics of the components  $x_1$  and  $x_2$ ,

$$\ddot{x}_i + (\omega_+ - \omega_-)\epsilon_{ij}\dot{x}_j + \omega_+\omega_-x_i = 0, \quad (2.13)$$

where  $\omega_{\pm}$  are defined in (2.3). This should be compared with the second order equations for the chiral modes,  $\ddot{X}_i^{\pm} + \omega_{\pm}^2 X_i^{\pm} = 0$ . The dynamics of  $x_1$  and  $x_2$  decouples only in the isotropic ( $\chi = 0$ ) case

$$\alpha_+ = \alpha_- = 1. \quad (2.14)$$

In terms of variables  $x_i$  and  $v_i$ , Lagrangian (2.2) takes, up to a total derivative, a non-chiral form

$$L = \mathcal{M} \left( \dot{x}_i v_i - \frac{\alpha_+ \alpha_-}{2R^2} x_i^2 - \frac{\mathcal{N}}{2\mathcal{M}} v_i^2 \right) + (\mu_- - \mu_+) \left( \alpha_+ \alpha_- \epsilon_{ij} x_i v_j - \frac{R^2}{2} \epsilon_{ij} v_i \dot{v}_j \right) + \frac{\mathcal{B}}{2} \epsilon_{ij} x_i \dot{x}_j, \quad (2.15)$$

where  $\mathcal{N} = R(\mu_+ \alpha_+ + \mu_- \alpha_-)$ . Note that  $\mathcal{M}$  and  $\mathcal{N}$  have units of mass, while  $\mathcal{B}$  does of a magnetic field. The term  $\epsilon_{ij} v_i \dot{v}_j$ , with coefficient proportional to  $(\mu_+ - \mu_-)$  in (2.15) is responsible for the coordinate non-commutativity (2.9). Physically it describes a magnetic like coupling for velocities. The terms  $\dot{x}_i v_i$  and  $\epsilon_{ij} x_i \dot{x}_j$ , with coefficients  $\mathcal{M}$  and  $\frac{1}{2}\mathcal{B}$ , correspond to the nontrivial brackets between  $x_i$  and  $v_i$ , and to the velocity non-commutativity, see the first and second relations in (2.10).

The equations of motion obtained by variation of (2.15) in  $x_i$  and  $v_i$ , are, respectively,

$$\mathcal{M} \left( \dot{v}_i + \frac{\alpha_+ \alpha_-}{R^2} x_i \right) = \epsilon_{ij} (\mathcal{B} \dot{x}_j + (\mu_- - \mu_+) \alpha_+ \alpha_- v_j), \quad (2.16)$$

$$R^2(\mu_+ - \mu_-) \left( \dot{v}_i + \frac{\alpha_+ \alpha_-}{R^2} x_i \right) = \epsilon_{ij} (\mathcal{M} \dot{x}_j - \mathcal{N} v_j). \quad (2.17)$$

If  $\mu_+ = \mu_-$ , and hence,  $\{x_i, x_j\} = 0$ , (2.17) takes the form (2.5), that is an algebraic equation for  $v_i$ . In this case  $v_1$  and  $v_2$  become auxiliary variables and can be eliminated by their equations of motion from (2.15). As a result, (2.15) turns into a regular, second order Lagrangian  $L(x, \dot{x})$ , that describes a usual Landau problem in the presence of additional isotropic harmonic potential term.

It is necessary to note that though for  $\mu_+ \neq \mu_-$  Eq. (2.5) also appears as a consequence of the system of equations (2.16) and (2.17), it is not produced by the variation of Lagrangian in  $v_i$  itself. If in this case we try to substitute  $v_i$ , using (2.5), in Lagrangian (2.15), we would get a higher derivative, nonequivalent Lagrangian, that generates the equations of motion different from (2.13).

In the isotropic case (2.14), system (2.15) is reduced to the exotic Newton-Hooke particle, which was constructed in [13] by the nonlinear realization method [16] accommodated to the space-time symmetries, see for example [17].

Let us show now that the case of the maximal anisotropy,

$$\chi = \varepsilon \frac{\pi}{4} : \quad \alpha_{\varepsilon} = 1, \quad \alpha_{-\varepsilon} = 0, \quad \varepsilon = +, -, \quad (2.18)$$

corresponds to the non-commutative Landau problem described by the Hamiltonian

$$H = \frac{1}{2m} \mathcal{P}_i^2, \quad (2.19)$$

symplectic structure

$$\{\mathcal{X}_i, \mathcal{X}_j\} = \frac{\theta}{1-\beta} \epsilon_{ij}, \quad \{\mathcal{X}_i, \mathcal{P}_i\} = \frac{1}{1-\beta} \delta_{ij}, \quad \{\mathcal{P}_i, \mathcal{P}_j\} = \frac{B}{1-\beta} \epsilon_{ij}, \quad (2.20)$$

and equations of motion

$$\dot{\mathcal{X}}_i = \frac{1}{m^*} \mathcal{P}_i, \quad \dot{\mathcal{P}}_i = \frac{B}{m^*} \epsilon_{ij} \mathcal{P}_j. \quad (2.21)$$

Here  $B$  is magnetic field,  $\beta = \theta B$ ,  $m^* = m(1 - \beta)$  plays the role of the effective mass, while  $\theta$  is a parameter, which at  $B = 0$  characterizes a non-commutativity of the coordinates and of the Galilean boosts of a free exotic particle [5, 15]. Lagrangian corresponding to (2.19) and (2.20) is given by

$$L_{NLP} = \mathcal{P}_i \dot{\mathcal{X}}_i - \frac{1}{2m} \mathcal{P}_i^2 + \frac{1}{2} \theta \epsilon_{ij} \mathcal{P}_i \dot{\mathcal{P}}_j + \frac{1}{2} B \epsilon_{ij} \mathcal{X}_i \dot{\mathcal{X}}_j. \quad (2.22)$$

Comparing (2.15) and (2.22), we find that in maximally anisotropic case (2.18) the former system reduces to the latter one under the following correspondence between the variables and parameters:

$$x_i = \mathcal{X}_i, \quad v_i = \frac{1}{m^*} \mathcal{P}_i, \quad (2.23)$$

$$\mu_\varepsilon = |B(1 - \theta B)|, \quad \mu_{-\varepsilon} = |B| \text{sgn}(1 - \theta B), \quad R = |\omega^{-1}|, \quad (2.24)$$

$$\varepsilon = \text{sgn}(B(\beta - 1)), \quad \omega = \frac{B}{m^*}. \quad (2.25)$$

Having this correspondence, and using transformation (2.8), we obtain the chiral form of Lagrangian (2.22) for the case  $\alpha_+ = 0$ ,  $\alpha_- = 1$  ( $\varepsilon = -1$ ),

$$L_{NLP}^{+-} = -\frac{B}{2} \epsilon_{ij} \dot{X}_i^+ X_j^+ - \frac{B(1 - \beta)}{2} \left( -\epsilon_{ij} \dot{X}_i^- X_j^- + \omega X_i^{-2} \right), \quad (2.26)$$

which generates equations of motion

$$\dot{X}_i^+ = 0, \quad \dot{X}_i^- - \omega \epsilon_{ij} X_j^- = 0. \quad (2.27)$$

In terms of the variables  $\mathcal{X}_i$  and  $\mathcal{P}_i$ , the chiral (normal) modes are given by

$$X_i^+ = \mathcal{X}_i + \frac{1}{m^* |\omega|} \epsilon_{ij} \mathcal{P}_j, \quad X_i^- = -\frac{1}{m^* |\omega|} \epsilon_{ij} \mathcal{P}_j. \quad (2.28)$$

The chiral mode  $X_i^+$  is an integral of motion not depending explicitly on time (cf. the chiral integrals of motion (3.2) in a generic case). It can be identified as a guiding center coordinate.

The case  $\alpha_+ = 1$ ,  $\alpha_- = 0$  ( $\varepsilon = +1$ ) can be obtained via obvious changes in correspondence with relations (2.24), (2.25). In this case the chiral mode  $X_i^-$  plays the role of the guiding center coordinate, while  $X_i^+$  has the same evolution law as the chiral mode  $X_i^-$  in the previous case  $\varepsilon = -1$ .

The flat limit

$$R \rightarrow \infty, \quad (\mu_+ - \mu_-) \rightarrow 0, \quad R\mu_+(\alpha_+ + \alpha_-) \rightarrow m, \quad R^2(\mu_+ - \mu_-) \rightarrow \theta m^2, \quad (2.29)$$

applied to (2.15), produces a free exotic particle,

$$L_\theta = m \left( \dot{x}_i v_i - \frac{1}{2} v_i^2 \right) + \frac{1}{2} \theta m^2 \epsilon_{ij} v_i \dot{v}_j, \quad (2.30)$$

that is described by the equations of motion  $\dot{x}_i = v_i$ ,  $\dot{v}_i = 0$ , and carries the two-fold centrally extended Galilei symmetry [5]. If, as in generic case (2.15), we try to substitute  $v_i$  using the equation  $v_i = \dot{x}_i$  produced by variation of (2.30) in  $x_i$ , we would get a nonequivalent higher derivative model [4] with additional, spin degrees of freedom, see [7, 8].

### 3 Symmetries: chiral picture

To identify the symmetries of our system, we proceed from the chiral Lagrangian (2.2). We integrate equations (2.3),

$$X_i^\pm(t) = \Delta_{ij}^\pm(t) X_j^\pm(0), \quad \Delta_{ij}^\pm(t) = \delta_{ij} \cos(\alpha_\pm t/R) \mp \epsilon_{ij} \sin(\alpha_\pm t/R), \quad (3.1)$$

and construct the integrals of motion,

$$\mathcal{J}_i^\pm \equiv R\mu_\pm \epsilon_{ij} X_j^\pm(0) = R\mu_\pm \epsilon_{ij} \Delta_{jk}^\pm(-t) X_k^\pm, \quad (3.2)$$

$$\mathcal{J}^\pm \equiv \pm \frac{\mu_\pm}{2} (X_i^\pm(0))^2 = \pm \frac{\mu_\pm}{2} (X_i^\pm)^2, \quad (3.3)$$

where  $X_i^\pm = X_i^\pm(t)$ . The quantities  $\mathcal{J}_i^\pm$  are the integrals of motion that include explicit dependence on time and satisfy the equation  $\frac{d}{dt} \mathcal{J}_i^\pm = \frac{\partial}{\partial t} \mathcal{J}_i^\pm + \{\mathcal{J}_i^\pm, H\} = 0$ , where

$$H = \frac{1}{2R} (\mu_+ \alpha_+ X_i^{+2} + \mu_- \alpha_- X_i^{-2}) \quad (3.4)$$

plays the role of the Hamiltonian. Unlike the linear in  $X_i^\pm$  integrals (3.2), the quadratic integrals (3.3) do not include explicit dependence on time.

Integrals (3.2) and (3.3) generate the algebra

$$\{\mathcal{J}^+, \mathcal{J}_i^+\} = \epsilon_{ij} \mathcal{J}_j^+, \quad \{\mathcal{J}_i^+, \mathcal{J}_j^+\} = Z^+ \epsilon_{ij}, \quad (3.5)$$

$$\{\mathcal{J}^-, \mathcal{J}_i^-\} = \epsilon_{ij} \mathcal{J}_j^-, \quad \{\mathcal{J}_i^-, \mathcal{J}_j^-\} = Z^- \epsilon_{ij}, \quad (3.6)$$

where  $Z^\pm = \mp R^2 \mu_\pm$  have a sense of central charges, and all other brackets are equal to zero. This is a chiral form of the (2+1)D exotic Newton-Hooke symmetry presented in the form of a direct sum of the two (1+1)D centrally extended Newton-Hooke algebras. The quadratic Casimirs of this algebra are

$$\mathcal{C}_\pm = \mathcal{J}_i^{\pm 2} + 2Z^\pm \mathcal{J}^\pm. \quad (3.7)$$

Let us stress that the exotic Newton-Hooke algebra (3.5), (3.6) in a generic case (2.6) has exactly the same form as in the particular isotropic case (2.14). For the latter case it was obtained in [13] by a contraction of the  $AdS_3$  algebra, with identification of the parameter  $R$  as the  $AdS_3$  radius. This chiral form of the ENH symmetry is rooted in the algebra isomorphism

$$AdS_3 \sim so(2, 2) \sim so(2, 1) \oplus so(2, 1) \sim AdS_2 \oplus AdS_2.$$

Integrals (3.2) and (3.3) generate the symmetry transformations of the chiral coordinates<sup>3</sup>,

$$\{X_i^+, \mathcal{J}_j^+\} = -R\Delta_{ij}^+(t), \quad \{X_i^-, \mathcal{J}_j^-\} = R\Delta_{ij}^-(t), \quad (3.8)$$

$$\{X_i^+, \mathcal{J}^+\} = -\epsilon_{ij} X_j^+, \quad \{X_i^-, \mathcal{J}^-\} = -\epsilon_{ij} X_j^-. \quad (3.9)$$

Due to the presence of the explicit dependence on time of the chiral integrals (3.2), symmetry transformations (3.8) are also time-dependent. Under them, Lagrangian (2.2) is quasi-invariant.

The Hamiltonian and the angular momentum are identified as linear combinations of the quadratic integrals, which generate the time translations and space rotations,

$$H = \frac{1}{R} (\alpha_+ \mathcal{J}^+ - \alpha_- \mathcal{J}^-), \quad J = \mathcal{J}^+ + \mathcal{J}^-. \quad (3.10)$$

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<sup>3</sup>We do not indicate explicitly infinitesimal transformation parameters.

This form of  $H$  and  $J$  is behind the anisotropy of dynamics and rotational symmetry of the system (2.2).

For the NLP case (2.18), one of the chiral generators,  $\mathcal{J}^+$  or  $\mathcal{J}^-$ , disappears from the Hamiltonian. As a result, the corresponding chiral mode has a trivial dynamics of the guiding center coordinate,  $\dot{X}^+ = 0$ , or  $\dot{X}_i^- = 0$ , see Eq. (2.27), and one of the two vectors (3.2) transforms into an integral of motion that does not depend explicitly on time.

Notice that in the excluded case  $\alpha_+ = -\alpha_-$ , the time translation,  $H$ , and rotation,  $J$ , generators would be (up to a multiplicative constant) the same, that prevents the introduction of the coordinate  $x_i$  and velocity  $v_i$  related by Eq. (2.5).

Let us discuss briefly symmetries in the quantum case.

We define the operators

$$a^- = \sqrt{\frac{|\mu_+|}{2}} (X_2^+ + iX_1^+), \quad a^+ = (a^-)^\dagger, \quad b^- = \sqrt{\frac{|\mu_-|}{2}} (X_1^- + iX_2^-), \quad b^+ = (b^-)^\dagger, \quad (3.11)$$

which obey the commutation relations  $[a^-, a^+] = \epsilon_+$ ,  $[b^-, b^+] = \epsilon_-$  and  $[a^\pm, b^\pm] = 0$ , where  $\epsilon_\pm = \text{sgn}(\mu_\pm)$ . For  $\mu_+ > 0$  ( $\mu_+ < 0$ ) and  $\mu_- > 0$  ( $\mu_- < 0$ ), the operators  $a^+$  ( $a^-$ ) and  $b^+$  ( $b^-$ ) are identified as creation oscillator operators. With the symmetrized ordering prescription, (3.3) and (3.10) give the Hamiltonian and the angular momentum operators in a form

$$RH = \epsilon_+ \alpha_+ a^+ a^- + \epsilon_- \alpha_- b^+ b^- + \frac{1}{2} (\epsilon_+ \alpha_+ + \epsilon_- \alpha_-), \quad (3.12)$$

$$J = \epsilon_+ a^+ a^- - \epsilon_- b^+ b^- + \frac{1}{2} (\epsilon_+ - \epsilon_-). \quad (3.13)$$

From here it follows that in the generic anisotropic case, like in particular cases of the ENH particle [13] and the NLP [15], our system can exhibit three different kinds of behavior in dependence on the values of the parameters  $\mu_\pm$ .

Since the eigenvalues of the number operators take non-negative integer values,  $n_a, n_b = 0, 1, \dots$ , we find that when  $\epsilon_+ = \epsilon_-$ ,  $J$  can take values of both signs. Its spectrum is unbounded. This case we call a *subcritical* phase. The spectrum of  $H$  in this phase is bounded from below (when  $\mu_+, \mu_- > 0$ ), or from above (for  $\mu_+, \mu_- < 0$ ).

When the signs of  $\mu_+$  and  $\mu_-$  are opposite, the spectrum of  $J$  is bounded from one side. This is a *supercritical* phase. In this phase the spectrum of  $H$  is unbounded, except the NLP case. The case of the NLP is special: its energy is bounded in both, sub- and super-critical phases, because one of the chiral modes has zero frequency, and does not contribute to the energy.

Yet, another phase corresponds to the case when one of the parameters  $\mu_+$  or  $\mu_-$  takes zero value. In such a phase one of the modes disappears from Lagrangian (taking a role of a pure gauge degree of freedom), and the system transforms into a one-dimensional oscillator, whose symmetry is described by the (1+1)D centrally extended Newton-Hooke algebra [13]. This is a *critical* phase that separates two other phases, and is characterized by the zero value of one of the central charges,  $Z^+$ , or  $Z^-$ . In the NLP it corresponds to the quantum Hall effect phase. We note here that in the critical phase, in turn, two different cases should be distinguished. When, say,  $\mu_- = 0$  and  $\alpha_+ = 1$ , the Hamiltonian is nontrivial and generates a rotation of the remaining chiral mode  $X_i^+$ , that coincides (up to a gauge shift) with  $x_i$ . When  $\mu_- = 0$  and  $\alpha_+ = 0$ , Hamiltonian is equal to zero, and the chiral mode has a trivial dynamics,  $X_i^+(t) = X_i^+(0)$  [for a discussion of the critical phase in the NLP, see ref. [15]]. In both cases,  $\alpha_+ = 1$  and  $\alpha_+ = 0$ ,  $x_i$  and  $v_i$  satisfy relation (2.5), but they are linearly dependent variables in correspondence with decreasing of the number of the physical degrees of freedom.

From the point of view of the dynamics and symmetries, as in the case of the usual planar anisotropic harmonic oscillator given by the second order Lagrangian, it is also necessary to distinguish special cases. As it follows from (2.8) and (3.1), the particle trajectory  $x_i(t)$  is closed only when  $\alpha_+/\alpha_-$  is rational. Behind this property, there is additional symmetry.

In the isotropic case (2.14), the ENH particle system is characterized by additional symmetry associated with the integrals  $a^+b^-$  and  $a^-b^+$ , which, like  $H$  and  $J$ , do not include explicit dependence on the time. In terms of the chiral modes, these are linear combinations of the Hermitian operators  $(X_1^+X_2^- + X_2^+X_1^-)$  and  $(X_2^+X_2^- - X_1^+X_1^-)$ . In sub- and super- critical phases, they together with angular momentum  $J$  generate the  $so(3)$  and  $so(2,1)$  symmetries, which are responsible for a finite and infinite degeneracy of the energy levels. This additional symmetry was discussed in detail in [13].

In the NLP, analogously, we have additional  $so(2,1)$  symmetry. If, say,  $\alpha_+ = 0$ ,  $\alpha_- = 1$ , and  $\mu_+ > 0$ , the  $so(2,1)$  generators are given by the quadratic integrals  $I_0 = \frac{1}{4}\{a^+, a^-\}$ ,  $I_+ = \frac{1}{2}a^{+2}$  and  $I_- = \frac{1}{2}a^{-2}$ ,

$$[I_0, I_{\pm}] = \pm I_{\pm}, \quad [I_+, I_-] = -2I_0. \quad (3.14)$$

All the energy levels are infinitely degenerate.

In the anisotropic case with  $\alpha_+/\alpha_- = p/q$ ,  $p, q = 1, 2, \dots$ ,  $p \neq q$ , the frequencies of the chiral modes are commensurable, and the system has additional integrals of motion  $j_+ = (a^+)^q(b^-)^p$  and  $j_- = (a^-)^q(b^+)^p$ . In this case the energy levels have finite, or infinite additional degeneracy, in dependence on whether we have a sub-, or super- critical phase. These integrals together with the angular momentum generate a nonlinear (polynomial) deformation of the  $so(3)$ , or  $so(2,1)$  algebra,

$$[J, j_{\pm}] = \pm(q+p)j_{\pm}, \quad (3.15)$$

$$[j_+, j_-] = \prod_{k=1}^q [a^+a^- + (1-k)\epsilon_+] \prod_{\ell=1}^p [b^+b^- + \ell\epsilon_-] - \prod_{k=1}^q [a^+a^- + k\epsilon_+] \prod_{\ell=1}^p [b^+b^- + (1-\ell)\epsilon_-], \quad (3.16)$$

where

$$a^+a^- = \frac{\epsilon_+}{(\alpha_+ + \alpha_-)} (RH + \alpha_-J) - \frac{1}{2}, \quad b^+b^- = \frac{\epsilon_-}{(\alpha_+ + \alpha_-)} (RH - \alpha_+J) - \frac{1}{2}, \quad (3.17)$$

in which the Hamiltonian plays a role of the central element,  $[H, J] = [H, j_{\pm}] = 0$ . This is completely analogous to the very well known property of a usual (non-exotic) planar anisotropic oscillator with commensurable frequencies, see [18, 19, 20, 21].

## 4 Exotic Newton-Hooke symmetry: space-time picture

Here we find the symmetry transformations in terms of the variables  $x_i$ , and  $v_i$  of the (2+1)D particle system, and their corresponding generators. In particular, we identify the integrals, which in a flat limit are transformed into commuting translations and non-commuting boosts generators, and find the algebra formed by them together with  $H$  and  $J$ .

To identify the translations and boosts generators, we note that because of the vector nature, they have to be linear combinations of the integrals  $\mathcal{J}_i^{\pm}$ . The transformations produced by  $\mathcal{J}_i^{\pm}$ , in correspondence with relations (3.8) and (2.8), are

$$\{x_i, \mathcal{J}_j^{\pm}\} = \mp R(\alpha_+ + \alpha_-)^{-1} \Delta_{ij}^{\pm}(t), \quad \{v_i, \mathcal{J}_j^{\pm}\} = \alpha_{\pm}(\alpha_+ + \alpha_-)^{-1} \epsilon_{ik} \Delta_{kj}^{\pm}(t), \quad (4.1)$$

where  $\Delta_{ij}^{\pm}(t)$  is defined in (3.1). Then, in order to recover the Galilean transformations in the flat limit (2.29),  $\{x_i, P_j\} = \delta_{ij}$ ,  $\{x_i, K_j\} = -\delta_{ij}t$ ,  $\{v_i, P_j\} = 0$ ,  $\{v_i, K_j\} = -\delta_{ij}$ , we obtain

$$P_i = \frac{1}{R} (\alpha_+ \mathcal{J}_i^- - \alpha_- \mathcal{J}_i^+), \quad K_i = -\epsilon_{ij} (\mathcal{J}_j^+ + \mathcal{J}_j^-). \quad (4.2)$$



Note that the relation (4.2) between  $P_i$ ,  $K_i$  and  $\mathcal{J}_i^\pm$  has a structure similar to that between  $H$ ,  $J$  and  $\mathcal{J}^\pm$ , see (3.10). The symmetry transformations generated by  $P_i$  and  $K_i$  can be computed by means of (4.1) and (4.2).

In correspondence with (2.5) and (2.13), the time translations symmetry transformations take here a form

$$\{x_i, H\} = v_i, \quad \{v_i, H\} = -\frac{\alpha_+ \alpha_-}{R^2} x_i - \frac{\alpha_+ - \alpha_-}{R} \epsilon_{ij} v_j, \quad (4.3)$$

where, in correspondence with (2.15),

$$H = \alpha_+ \alpha_- \left( \frac{\mathcal{M}}{2R^2} x_i^2 + (\mu_+ - \mu_-) \epsilon_{ij} x_i v_j \right) + \frac{\mathcal{N}}{2} v_i^2. \quad (4.4)$$

The second term in the transformation law for the velocity is proportional to the rotation symmetry transformation  $\{v_i, J\} = -\epsilon_{ij} v_j$ , and disappears in the isotropic case. It is worth to note that the structure of the angular momentum,

$$J = \frac{\mathcal{B}}{2} x_i^2 + \frac{1}{2} R^2 (\mu_+ - \mu_-) v_i^2 + \mathcal{M} \epsilon_{ij} x_i v_j, \quad (4.5)$$

reproduces the structure of the symplectic two-form (2.12).

The symmetry algebra generated by  $H$ ,  $J$ ,  $P_i$  and  $K_i$  is

$$\{K_i, K_j\} = -\tilde{Z} \epsilon_{ij}, \quad \{P_i, P_j\} = -\frac{1}{R^2} \left( R(\alpha_+ - \alpha_-) Z + \alpha_+ \alpha_- \tilde{Z} \right) \epsilon_{ij}, \quad \{K_i, P_j\} = Z \delta_{ij}, \quad (4.6)$$

$$\{K_i, J\} = -\epsilon_{ij} K_j, \quad \{P_i, J\} = -\epsilon_{ij} P_j, \quad \{H, J\} = 0, \quad (4.7)$$

$$\{K_i, H\} = P_i + \frac{(\alpha_+ - \alpha_-)}{R} \epsilon_{ij} K_j, \quad \{P_i, H\} = -\frac{\alpha_+ \alpha_-}{R^2} K_i, \quad (4.8)$$

where

$$Z = (\alpha_+ Z^- - \alpha_- Z^+) R^{-1} = \mathcal{M}, \quad \tilde{Z} = -(Z^+ + Z^-) = R^2 (\mu_+ - \mu_-), \quad (4.9)$$

and  $\mathcal{M}$  is defined in (2.11). Casimirs (3.7) take here an equivalent form

$$\mathcal{C}_\pm = \left( P_i \mp \frac{\alpha_\pm}{R} \epsilon_{ij} K_j \right)^2 - 2 \left( Z \pm \frac{\alpha_\pm}{R} \tilde{Z} \right) \left( H \pm \frac{\alpha_\mp}{R} J \right). \quad (4.10)$$

When  $\mu_+ = \mu_-$ , the central charge  $\tilde{Z}$  takes zero value, and the Galilean boosts commute. It is exactly the same case when the coordinates of the particle are commutative, see (2.9). Analogously, the commutativity of the boosts and translations takes place when another central charge disappears,  $Z = \mathcal{M} = 0$ . In this case the coordinate  $x_i$  and velocity  $v_i$  commute, see the first relation in (2.10).

For the particular case (2.18) of the non-commutative Landau problem the explicit form of the algebra generated by  $H$ ,  $J$ ,  $P_i$  and  $K_i$  can be obtained from (4.6)–(4.10) by means of the correspondence relations (2.23)–(2.25). We only note that the translation generator is reduced here for the conserved chiral mode identified with the guiding center coordinate, see (4.2). It generates usual time-independent translations,  $\delta x_i = \delta a_i$ ,  $\delta v_i = 0$ , under which Lagrangian (2.22) is quasi-invariant.

In the generic case, the generators  $H$ ,  $J$ ,  $P_i$  and  $K_i$ , and the central charges  $Z$  and  $\tilde{Z}$  are linear combinations of the chiral integrals  $\mathcal{J}^\pm$  and  $\mathcal{J}_i^\pm$  and central charges  $Z^+$  and  $Z^-$ , see Eqs. (3.10), (4.2) and (4.9). Hence, there exists a linear relation between the space-time, non-chiral symmetry generators of the exotic anisotropic harmonic oscillator characterized by the parameters  $\alpha_\pm(\chi)$ ,

and the space-time generators of the exotic Newton-Hooke symmetry which corresponds to the symmetric case (2.14). Explicitly, we have

$$\begin{pmatrix} RH \\ J \end{pmatrix}_{\alpha_+, \alpha_-} = \begin{pmatrix} \mathcal{A}_+ & \mathcal{A}_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} RH \\ J \end{pmatrix}_{\alpha_+ = \alpha_- = 1}, \quad (4.11)$$

$$\begin{pmatrix} RP_i \\ K_i \end{pmatrix}_{\alpha_+, \alpha_-} = \begin{pmatrix} \mathcal{A}_+ \delta_{ij} & \mathcal{A}_- \epsilon_{ij} \\ 0 & \delta_{ij} \end{pmatrix} \begin{pmatrix} RP_j \\ K_j \end{pmatrix}_{\alpha_+ = \alpha_- = 1}, \quad (4.12)$$

$$\begin{pmatrix} RZ \\ \tilde{Z} \end{pmatrix}_{\alpha_+, \alpha_-} = \begin{pmatrix} \mathcal{A}_+ & -\mathcal{A}_- \\ 0 & 1 \end{pmatrix} \begin{pmatrix} RZ \\ \tilde{Z} \end{pmatrix}_{\alpha_+ = \alpha_- = 1}, \quad (4.13)$$

where  $\mathcal{A}_\pm = \frac{1}{2}(\alpha_+ \pm \alpha_-)$ .

These relations mean that the general case of the exotic anisotropic harmonic oscillator, including the NLP system as a particular case, is described, in fact, by the same non-chiral, space-time form of the (2+1)D exotic Newton-Hooke symmetry, as the exotic isotropic oscillator does. On the other hand, with making use of (4.11)–(4.13), the generators and central charges of the generic anisotropic case can also be presented as linear combinations of the generators and central charges of the non-commutative Landau problem.

## 5 Discussion and concluding remarks

We have showed that the planar exotic anisotropic harmonic oscillator is characterized by exactly the same chiral form of the (2+1)D exotic Newton-Hooke symmetry algebra (3.5), (3.6) as in the isotropic case. The anisotropy reveals itself only in the symmetry transformations law, see Eqs. (3.8) and (3.1), that is associated with the anisotropy of the Hamiltonian structure (3.10). In the space-time picture the anisotropy reveals itself both in the symmetry transformations, and in the structure of the symmetry algebra. However, unlike the usual planar anisotropic harmonic oscillator given by the second order Lagrangian, anisotropy does not violate the rotation symmetry. It only mixes the time translations with the rotations, and the space translations with the boosts transformations, see Eqs. (4.11), (4.12). The presence of the two independent parameters  $\mu_+$  and  $\mu_-$  is behind the non-commutativity nature of the particle coordinate (2.9) in a generic case of the model.

The translations,  $P_i$ , and the boosts,  $K_i$ , generators of the exotic Newton-Hooke symmetry are the linear combinations of the first order in the chiral variables  $X_i^\pm$  integrals (3.2), that include explicit dependence on time. On the contrary, the  $H$  and  $J$  are the linear combinations of the explicitly time-independent, quadratic in (3.2), integrals (3.3). The additional symmetries of the isotropic and the NLP special cases, discussed at the end of Section 3, are also generated by the explicitly time-independent integrals, which are quadratic in the integrals  $\mathcal{J}_i^\pm$ . If we supply these four explicitly time-independent quadratic integrals with other six, explicitly time-dependent quadratic in  $\mathcal{J}_i^\pm$  integrals of motion, we would get a more broad, the  $AdS_4 \sim so(3,2) \sim sp(4)$  symmetry as a symmetry of the system. With respect to the ten  $so(3,2)$  generators, which are certain linear combinations of the quadratic quantities  $L_a L_b$ ,  $a = 1, \dots, 4$ ,  $L_a = (\mathcal{J}_i^+, \mathcal{J}_j^-)$ , the integrals  $\mathcal{J}_i^\pm$  form a Majorana spinor [22, 23]. From the point of view of the  $so(3,2)$  algebra, the Hamiltonian  $H$  and the angular momentum  $J$  of the exotic anisotropic harmonic oscillator are just linearly independent combinations of the one of spatial rotation generators in an abstract (3+2)D space-time, and of the generator of rotations in the plane of the two time-like coordinates in that space-time, see [23]. We notice here that the system (2.2) can be related to the gauge fixed version of the  $Sp(4)$  gauge invariant particle mechanics model [24].

To conclude, since there are some indications on the possible close relation of the ENH particle to the physics of the BTZ black hole [13], it would be interesting to clarify whether the exotic anisotropic harmonic oscillator, and the non-commutative Landau problem as its particular case, could be related to the 3D gravity physics. A close relation between the usual Landau problem and a family of Gödel-type solutions in M-theory and 3+1 General Relativity was pointed out recently in [25, 26].

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